MATH20132 Calculus of Several Variables. 2020-21

Solutions to Problems 7: Graphs, level sets, parametric sets, Implicit & Inverse functions

Inverses

1. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \exp(x)\cos(y) \\ \exp(x)\sin(y) \end{pmatrix},$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$.

i. Prove that the linear map $d\mathbf{f}_\mathbf{a} : \mathbb{R}^2 \to \mathbb{R}^2$ has an inverse for all $\mathbf{a} \in \mathbb{R}^2$ but that $\mathbf{f}$ does not have an inverse.

ii. Let $\mathbf{f}_0 : U = \{ (x, y)^T \in \mathbb{R}^2 \mid -\pi < y < \pi\} \to \mathbb{R}^2$ be the restriction of $\mathbf{f}$. Prove that $\mathbf{f}_0$ is an injection.

iii. If $\mathbf{g} : \mathbf{f}_0(U) \to U$ is inverse of $\mathbf{f}_0$ find the Jacobian matrix of $\mathbf{g}$ at $\mathbf{b} = \mathbf{f}_0(\mathbf{a})$.

Hint: Use the Chain Rule.

Solution i. A linear map has an inverse iff the associated matrix is invertible. The associated matrix of $d\mathbf{f}_\mathbf{a}$ is the Jacobian matrix

$$J\mathbf{f}(\mathbf{x}) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}.$$

This has determinant $e^{2x}\left(\cos^2 y + \sin^2 y\right) = e^{2x} \neq 0$ for all $\mathbf{x} \in \mathbb{R}^2$. Thus, for any $\mathbf{a} \in \mathbb{R}^2$, $J\mathbf{f}(\mathbf{a})$ is invertible, in which case $d\mathbf{f}_\mathbf{a}$ has an inverse.

Because

$$\mathbf{f}\left(\begin{pmatrix} x \\ y + 2\pi \end{pmatrix}\right) = \mathbf{f}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

the function $\mathbf{f}$ is not an injection and thus does not have an inverse.

ii. To see that $\mathbf{f}_0$ is an injection assume $\mathbf{f}_0(\mathbf{x}_1) = \mathbf{f}_0(\mathbf{x}_2)$ for some $\mathbf{x}_1 = (x_1, y_1)^T$, $\mathbf{x}_2 = (x_2, y_2)^T \in U$. Then

$$e^{x_1} = |\mathbf{f}_0(\mathbf{x}_1)| = |\mathbf{f}_0(\mathbf{x}_2)| = e^{x_2},$$

so $x_1 = x_2 = x$, say. Next

$$\begin{pmatrix} e^x \cos y_1 \\ e^x \sin y_1 \end{pmatrix} = \mathbf{f}_0\left(\mathbf{x}_1\right) = \mathbf{f}_0\left(\mathbf{x}_2\right) = \begin{pmatrix} e^x \cos y_2 \\ e^x \sin y_2 \end{pmatrix}.$$

so $\cos y_1 = \cos y_2$ and $\sin y_1 = \sin y_2$. The only solution of these in the restricted range $-\pi < y_1, y_2 < \pi$ is $y_1 = y_2$. Hence

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \mathbf{x}_2,$$

and so $\mathbf{f}_0$ is injective.

iii. Since $\mathbf{g} \circ \mathbf{f}_0 = 1_U$ the Chain Rule gives $J\mathbf{g}(\mathbf{f}_0(\mathbf{a}))\, J\mathbf{f}_0(\mathbf{a}) = I_2$, so, as noted above,

$$\begin{aligned} J\mathbf{g}(\mathbf{b}) &= J\mathbf{f}_0(\mathbf{a})^{-1} = \begin{pmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{pmatrix}^{-1} \\ &= e^{-2a} \begin{pmatrix} e^a \cos b & e^a \sin b \\ -e^a \sin b & e^a \cos b \end{pmatrix} = \begin{pmatrix} e^{-a} \cos b & e^{-a} \sin b \\ -e^{-a} \sin b & e^{-a} \cos b \end{pmatrix}. \end{aligned}$$

2. State Inverse Function Theorem:

i. Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{x} \mapsto \left(x^2 y, xy^2\right)^T$.

a. Prove that $\mathbf{f}$ has a local inverse at $\mathbf{a} = (a, b)^T$ (i.e. has an differentiable inverse when restricted to some open set containing $\mathbf{a}$) if and only if $ab \neq 0$.

b. Find the Jacobian matrix $J\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big)$ of the local inverse $\mathbf{g} = \mathbf{f}^{-1}$ at $\mathbf{f}(\mathbf{a})$, when it exists.

ii. Define the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{x} \mapsto (yz, xz, xy)^T$.

a. Prove that $\mathbf{f}$ has a local inverse at $\mathbf{a} = (a, b, c)^T$ (i.e. has an differentiable inverse when restricted to an open set containing $\mathbf{a}$) if and only if $abc \neq 0$.

b. Find the Jacobian matrix $J\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big)$ of the local inverse $\mathbf{g} = \mathbf{f}^{-1}$ at $\mathbf{f}(\mathbf{a})$, when it exists.

Solution From the notes the Inverse Function Theorem: *Suppose that* $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *is a* $\mathcal{C}^1$*-function such that for some* $\mathbf{a} \in U$ *the Jacobian matrix* $J\mathbf{f}(\mathbf{a})$ *is of full rank then* $\mathbf{f}$ *is locally invertible. That is, there exists an open set* $V : \mathbf{a} \in V \subseteq U$, *such that*

- $\mathbf{f} : V \to \mathbf{f}(V)$ *is a bijection,*
- $\mathbf{f}(V)$ *is an open subset of* $\mathbb{R}^n$,
- *the inverse function* $\mathbf{g} = \mathbf{f}^{-1} : \mathbf{f}(V) \to V$ *is* C^1 *and* $d\mathbf{g}_{\mathbf{b}} = d\mathbf{f}_{\mathbf{a}}^{-1}$, *or* $J\mathbf{g}(\mathbf{b}) = J\mathbf{f}(\mathbf{a})^{-1}$, *where* $\mathbf{b} = \mathbf{f}(\mathbf{a})$.

i.a. The Jacobian matrix at the general point $\mathbf{a} = (a, b)^T$ is

$$J\mathbf{f}(\mathbf{a}) = \begin{pmatrix} 2ab & a^2 \\ b^2 & 2ab \end{pmatrix}.$$

Then, by the Inverse Function Theorem, $\mathbf{f}$ is locally invertible at $\mathbf{a}$ iff $J\mathbf{f}(\mathbf{a})$ is invertible iff $\det J\mathbf{f}(\mathbf{a}) \neq 0$ iff $3a^2b^2 \neq 0$ iff $ab \neq 0$.

b. If $\mathbf{g}$ is the local inverse to $\mathbf{f}$ then $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ and the Chain Rule gives $J\mathbf{g}(\mathbf{f}(\mathbf{x}))\, J\mathbf{f}(\mathbf{x}) = I$. Choose $\mathbf{x} = \mathbf{a}$ so $J\mathbf{g}(\mathbf{f}(\mathbf{a}))\, J\mathbf{f}(\mathbf{a}) = I$. Thus

$$J\mathbf{g}(\mathbf{f}(\mathbf{a})) = \frac{1}{3a^2b^2} \begin{pmatrix} 2ab & -a^2 \\ -b^2 & 2ab \end{pmatrix}.$$

ii. a. The Jacobian matrix of $\mathbf{f}$ is

$$J\mathbf{f}(\mathbf{a}) = \begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \quad \text{and} \quad \det J\mathbf{f}(\mathbf{a}) = abc.$$

Thus $J\mathbf{f}(\mathbf{a})$ is invertible if, and only if, $abc \neq 0$.

In particular $abc \neq 0$ implies $J\mathbf{f}(\mathbf{a})$ is invertible which in turn implies $\mathbf{f}$ is locally invertible by the Inverse Function Theorem.

Conversely if $\mathbf{f}$ is locally invertible there exists $\mathbf{g}$ such that $\mathbf{g} \circ \mathbf{f}$ is the identity when the Chain Rule implies $J\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big) J\mathbf{f}(\mathbf{a}) = I$, so $J\mathbf{f}(\mathbf{a})$ is invertible which implies $abc \neq 0$.

Hence $\mathbf{f}$ is locally invertible if, and only if, $abc \neq 0$.

b. For the inverse $\mathbf{g}$, from $J\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big) J\mathbf{f}(\mathbf{a}) = I$ above we get

$$J\mathbf{g}\big(\mathbf{f}(\mathbf{a})\big) = J\mathbf{f}(\mathbf{a})^{-1} = \frac{1}{2abc} \begin{pmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{pmatrix}.$$

You can find this inverse by using row operations or, perhaps, the adjoint matrix (the transpose of the cofactor matrix).

3. Proof of the Inverse Function Theorem Assume that $\mathbf{f} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function such that at $\mathbf{a} \in U$ the Jacobian matrix $J\mathbf{f}(\mathbf{a})$ is of full-rank. Prove that the Inverse Function Theorem follows by applying the Implicit Function Theorem to the function $\mathbf{h} : \mathbb{R}^n \times U \subseteq \mathbb{R}^{2n} \to \mathbb{R}^n$ defined by

$$\mathbf{h}\left(\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right) = \mathbf{x} - \mathbf{f}(\mathbf{y}), \quad \text{where} \quad \mathbf{x} \in \mathbb{R}^n,\, \mathbf{y} \in U.$$

Hint: The important observation is that by definition of $\mathbf{h}$,

$$\mathbf{h}\left(\begin{pmatrix}\mathbf{f}(\mathbf{a})\\ \mathbf{a}\end{pmatrix}\right) = \mathbf{f}(\mathbf{a}) - \mathbf{f}(\mathbf{a}) = \mathbf{0}.$$

So, setting

$$\mathbf{p} = \begin{pmatrix}\mathbf{f}(\mathbf{a})\\ \mathbf{a}\end{pmatrix},$$

we have $\mathbf{h}(\mathbf{p}) = \mathbf{0}$ as required for the Implicit Function Theorem. To deduce anything from that Theorem it is required that $J\mathbf{h}(\mathbf{p})$ is of full rank. What is $J\mathbf{h}(\mathbf{p})$?

Solution Follow the hint and define $\mathbf{h} : \mathbb{R}^n \times U \subseteq \mathbb{R}^{2n} \to \mathbb{R}^n$ by

$$\mathbf{h}\left(\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right) = \mathbf{x} - \mathbf{f}(\mathbf{y}).$$

Given $\mathbf{w} \in \mathbb{R}^n \times U$ write it as $\mathbf{w} = \left(\mathbf{x}^T, \mathbf{y}^T\right)^T$ with $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in U$. The Jacobian matrix of $\mathbf{h}$ at $\mathbf{w}$ is then

$$J\mathbf{h}(\mathbf{w}) = (I_n \,|\, -J\mathbf{f}(\mathbf{y})).$$

Let

$$\mathbf{p} = \begin{pmatrix}\mathbf{f}(\mathbf{a})\\ \mathbf{a}\end{pmatrix} \in \mathbb{R}^n \times U,$$

with $\mathbf{a}$ the point at which $J\mathbf{f}(\mathbf{a})$ is non-singular. Then $J\mathbf{h}(\mathbf{p}) = (I_n \,|\, -J\mathbf{f}(\mathbf{a}))$ and, since $J\mathbf{f}(\mathbf{a})$ is non-singular, the last n columns of $J\mathbf{f}(\mathbf{p})$ are linearly independent.

Hence, by the Implicit Function Theorem, (and no permutation of coordinates is required), there exists

- an open set $A \subseteq \mathbb{R}^n$ containing $\mathbf{f}(\mathbf{a})$,
- a C^1-function $\phi : A \to \mathbb{R}^n$ and
- an open set $B \subseteq \mathbb{R}^n \times U$ containing $\mathbf{p}$ such for $\left(\mathbf{x}^T, \mathbf{y}^T\right)^T \in B$, $(\mathbf{x}, \mathbf{y} \in \mathbb{R}^n)$,

$$\mathbf{h}\left(\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right) = \mathbf{0} \text{ if, and only if, } \quad \mathbf{x} \in A \text{ and } \mathbf{y} = \phi(\mathbf{x}).$$

Note that the set B can be written as $A \times C$ for some open set $C \subseteq \mathbb{R}^n$. Thus we have

$$\mathbf{x} = \mathbf{f}(\mathbf{y}) \iff \mathbf{x} - \mathbf{f}(\mathbf{y}) = \mathbf{0} \iff \mathbf{h}\left(\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right) = \mathbf{0} \iff \mathbf{y} = \phi(\mathbf{x}).$$

This gives the existence of a C^1-inverse.

Image or Parametric sets are locally graphs.

4. Show that the following Image sets are locally graphs around the point given.

i. $\left\{((4+2\cos t)\cos s, (4+2\cos t)\sin s, 2\sin t)^T : s, t \in \mathbb{R}\right\}$ with $\mathbf{q} = (3\pi/4, \pi/4)^T$,

ii. $\left\{(xy^2,\, x^2+y,\, x^3-y^2,\, y^2)^T : x, y \in \mathbb{R}\right\}$, with $\mathbf{q} = (-1, 2)^T$,

iii. $\left\{(\cos t,\, \sin t,\, t)^T : t \in \mathbb{R}\right\}$ with $q = 3\pi$.

Solution To show that the image set of $\mathbf{F} : \mathbb{R}^r \to \mathbb{R}^n$ is locally a graph we apply a Corollary of the Inverse Function. This says that if the rows 1 to r of $J\mathbf{F}(\mathbf{q})$ are linearly independent then the image set is the graph of some function $\boldsymbol{\phi} : W \subseteq \mathbb{R}^r \to \mathbb{R}^{n-r}$ with $\mathbf{q} \in W$.

i.

$$J\mathbf{F}(\mathbf{q}) = \left.\begin{pmatrix} -(4+2\cos t)\sin s & -2\sin t\cos s \\ (4+2\cos t)\cos s & -2\sin t\sin s \\ 0 & 2\cos t \end{pmatrix}\right|_{\mathbf{s}=\mathbf{q}} = \begin{pmatrix} -\left(4+\sqrt{2}\right)/\sqrt{2} & 1 \\ -\left(4+\sqrt{2}\right)/\sqrt{2} & -1 \\ 0 & \sqrt{2} \end{pmatrix}$$

The first two rows are linearly independent and so the image set can be written as a graph of the first two variables in some open set around $\mathbf{q}$.

ii.

$$J\mathbf{F}(\mathbf{q}) = \begin{pmatrix} y^2 & 2xy \\ 2x & 1 \\ 3x^2 & -2y \\ 0 & 2y \end{pmatrix}_{\mathbf{x}=\mathbf{q}} = \begin{pmatrix} 4 & -4 \\ -2 & 1 \\ 3 & -4 \\ 0 & 4 \end{pmatrix}.$$

The first two rows are linearly independent and so the image set can be written as a graph of the first two variables in some open set around $\mathbf{q}$.

iii. This is already a graph. Usually a graph is written with the variables first followed by the functions of the variables. Here we have a permutation of this, functions first variable last.

Higher order derivatives

5. Return to Question 7 on Sheet 6. We showed that for the level set of $(x, y, u, v)^T \in \mathbb{R}^4$ satisfying

$$\begin{aligned} x^2 + y^2 + 2uv &= 4 \\ x^3 + y^3 + u^3 - v^3 &= 0, \end{aligned}$$

there exists an open subset of $\mathbb{R}^4$ containing the solution $\mathbf{p} = (-1, 1, 1, 1)$ in which the u and v can be given as functions of x and y, with $(x, y)^T$ in some open subset of $\mathbb{R}^2$ containing the point $\mathbf{q} = (-1, 1)^T$. Find the second order derivatives of u and v.

A purpose of Question 7 was to highlight the fact that when conditions are satisfied the Implicit Function Theorem ensures that functions exist, but gives no further information about them. Nonetheless their derivatives can be found. In this question we continue to find their second derivatives.

Advice for Exams *Know how to take second derivatives. Many students failed to calculate correctly second derivatives in the exam. Make sure that the exam is not the first time you attempt a question such as this one.*

Solution Write the level set $\mathbf{f}^{-1}(\mathbf{0})$ as the system

$$\begin{aligned} x^2 + y^2 + 2uv &= 4 \\ x^3 + y^3 + u^3 - v^3 &= 0. \end{aligned} \tag{1}$$

By differentiating w.r.t. x and w.r.t y we found in Question 7 on Sheet 6 that

$$\frac{\partial u}{\partial x}(\mathbf{q}) = 0 \quad \frac{\partial v}{\partial x}(\mathbf{q}) = 1, \quad \frac{\partial u}{\partial y}(\mathbf{q}) = -1 \quad \text{and} \quad \frac{\partial v}{\partial y}(\mathbf{q}) = 0. \tag{2}$$

Differentiate the system (1) twice, w.r.t x both times to get

$$\begin{aligned} 2 + 2\frac{\partial^2 u}{\partial x^2}v + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\frac{\partial^2 v}{\partial x^2}u + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} &= 0 \\ 6x + 6u\left(\frac{\partial u}{\partial x}\right)^2 + 3u^2\frac{\partial^2 u}{\partial x^2} - 6v\left(\frac{\partial v}{\partial x}\right)^2 - 3v^2\frac{\partial^2 v}{\partial x^2} &= 0. \end{aligned}$$

Then at $\mathbf{p}$, using (2), we get the simpler system

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = -1 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} = 4.$$

Solve these to give

$$\frac{\partial^2 u}{\partial x^2} = \frac{3}{2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = -\frac{5}{2},$$

where the derivatives are evaluated at $\mathbf{q}$.

Return and differentiate (1) first w.r.t x and then w.r.t. y to get

$$\begin{aligned} 2v\frac{\partial^2 u}{\partial y \partial x} + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + 2u\frac{\partial^2 v}{\partial y \partial x} &= 0, \\ 6u\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + 3u^2\frac{\partial^2 u}{\partial y \partial x} - 6v^2\frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - 3v^2\frac{\partial^2 v}{\partial y \partial x} &= 0. \end{aligned}$$

At $\mathbf{p}$, again using (2), this becomes

$$\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y \partial x} = 1 \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 v}{\partial y \partial x} = 0.$$

Hence

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y \partial x} = \frac{1}{2}.$$

I do not give the details for the mixed derivative in the reversed order, but you can check that $\partial^2 u/\partial x \partial y = \partial^2 u/\partial y \partial x$ and the same for v. It can, in fact, be shown that the order is immaterial whenever $\mathbf{f}$ is C^2.

Finally, differentiate the system w.r.t y twice to get

$$\begin{aligned} 2 + 2v\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + 2u\frac{\partial^2 v}{\partial y^2} &= 0 \\ 6y + 6u\left(\frac{\partial u}{\partial y}\right)^2 + 3u^2\frac{\partial^2 u}{\partial y^2} - 6v\left(\frac{\partial v}{\partial y}\right)^2 - 3v^2\frac{\partial^2 v}{\partial y^2} &= 0. \end{aligned}$$

At $\mathbf{p}$ this reduces to

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} = -1 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y^2} = -4.$$

Hence

$$\frac{\partial^2 u}{\partial y^2} = -\frac{5}{2} \quad \text{and} \quad \frac{\partial^2 v}{\partial y^2} = \frac{3}{2}.$$

Best Affine Approximations.

Recall that the Best Affine Approximation to a function $\mathbf{f}$ *at a point* $\mathbf{a}$ *is given by*

$$\mathbf{f}(\mathbf{a}) + d\mathbf{f}_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) = \mathbf{f}(\mathbf{a}) + J\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})\,.$$

6. Write down the Best Affine Approximation to

i. $f(\mathbf{x}) = x\,(x+y)$ at $\mathbf{a} = (2,-1)^T$, and what value does the approximation give at $\mathbf{a}' = (2.1,-0.9)^T$?

ii. $f(\mathbf{x}) = xy + yz + xz$ at $\mathbf{a} = (-1,-1,4)^T$, and what value does the approximation give at $\mathbf{a}' = (-0.9,-1.1,4.1)^T$?

iii.

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} xy^2 \\ x^2 y \end{pmatrix}$$

at $\mathbf{a} = (2,-3)^T$, and what value does the approximation give at $\mathbf{a}' = (1.9,-3.1)^T$?

Hint These functions have been seen previously on Sheet 4.

Solution i. As seen in Question 1 Sheet 4, the Fréchet derivative is $df_{\mathbf{a}}(\mathbf{t}) = (2\alpha+\beta)\,s + \alpha t$ for general $\mathbf{a} = (\alpha,\beta)^T$ and $\mathbf{t} = (s,t)^T$. With the specific $\mathbf{a} = (2,-1)^T$ this becomes $df_{\mathbf{a}}(\mathbf{t}) = 3s + 2t$. Then, with $\mathbf{t} = \mathbf{x} - \mathbf{a}$,

$$df_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) = 3\,(x-2) + 2\,(y+1) = 3x + 2y - 4,$$

where $\mathbf{x} = (x,y)^T$. Thus, the Best Affine Approximation is

$$f\,(\mathbf{a}) + 3x + 2y - 4 = 3x + 2y - 2.$$

At $\mathbf{a}' = (2.1,-0.9)^T$ the approximation is 2.5. The actual value is 2.52.

ii. As seen in Question 2 Sheet 4, the derivative is

$$df_{\mathbf{a}}(\mathbf{t}) = (\beta+\gamma)\, s + (\alpha+\gamma)\, t + (\alpha+\beta)\, u$$

for general $\mathbf{a} = (\alpha, \beta, \gamma)^T$ and $\mathbf{t} = (s, t, u)^T$. With the specific $\mathbf{a} = (-1, -1, 4)^T$ this becomes $df_{\mathbf{a}}(\mathbf{t}) = 3s + 3t - 2u$. Thus, the Best Affine Approximation is

$$f(\mathbf{a}) + 3\,(x+1) + 3\,(y+1) - 2\,(z-4) = 3x + 3y - 2z + 7.$$

At $\mathbf{a}' = (-0.9, -1.1, 4.1)^T$ the approximation is -7.2. The actual value is -7.21.

iii. As seen in Question 4 Sheet 4, the derivative is

$$d\mathbf{f}_{\mathbf{a}}(\mathbf{t}) = \begin{pmatrix} \beta^2 s + 2\alpha\beta t \\ 2\alpha\beta s + \alpha^2 t \end{pmatrix}.$$

With the specific $\mathbf{a} = (2, -3)^T$ this becomes

$$d\mathbf{f}_{\mathbf{a}}(\mathbf{t}) = \begin{pmatrix} 9s - 12t \\ -12s + 4t \end{pmatrix}.$$

Thus, the Best Affine Approximation is

$$\mathbf{f}\,(\mathbf{a}) + \begin{pmatrix} 9\,(x-2) - 12\,(y+3) \\ -12\,(x-2) + 4\,(y+3) \end{pmatrix} = \begin{pmatrix} 9x - 12y - 36 \\ -12x + 4y + 24 \end{pmatrix}.$$

At $\mathbf{a}' = (1.9, -3.1)^T$ the approximation is $\begin{pmatrix} 18.3 & -11.2 \end{pmatrix}^T$. The actual value is $\begin{pmatrix} 18.259 & -11.191 \end{pmatrix}^T$.

There is a form of Taylor's Theorem for scalar-valued functions of several variables. This can be used to estimate the error between the Best Affine Approximation and the original function. See my web site for (non-examinable) notes on this.

7. Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{f}(\mathbf{x}) = (x^3 - 2xy^2,\ x+y)^T$. Show that $\mathbf{f}$ locally invertible at $\mathbf{a} = (1, -1)^T$.

What is the Best Affine Approximation to the **inverse** function near $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (-1,\ 0)^T$?

What approximation does this give to $\mathbf{f}^{-1}\Big((-0.9,\ 0.1)^T\Big)$?

Solution The Jacobian matrix is

$$Jf(\mathbf{a}) = \begin{pmatrix} 3x^2-2y^2 & -4xy \\ 1 & 1 \end{pmatrix}_{(1,-1)^T} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

The matrix is of full-rank and so, by the Inverse Function Theorem, $\mathbf{f}$ is invertible in some open set containing $\mathbf{a}$ (i.e. it is locally invertible). From the Chain Rule applied to $\mathbf{f} \circ \mathbf{f}^{-1} = \mathbf{id}$ we deduce that the Jacobian of the inverse is the inverse of the Jacobian. That is, with $\mathbf{b} = \mathbf{f}(\mathbf{a})$, the Jacobian of the inverse is

$$J\mathbf{f}^{-1}(\mathbf{b}) = J\mathbf{f}(\mathbf{a})^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}.$$

Then the Best Affine Approximation to $\mathbf{f}^{-1}$ near $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (-1,0)^T$ is, as a function of $\mathbf{u} = (u,v)^T \in \mathbb{R}^2$,

$$\begin{aligned} \mathbf{f}^{-1}(\mathbf{b}) + J\mathbf{f}^{-1}(\mathbf{b})(\mathbf{u}-\mathbf{b}) &= \mathbf{a} + J\mathbf{f}(\mathbf{a})^{-1}(\mathbf{u}-\mathbf{f}(\mathbf{a})) \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u+1 \\ v \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 2-u+4v \\ -2+u-v \end{pmatrix}. \end{aligned}$$

The approximation this gives to $\mathbf{f}^{-1}\left((-0.9,\,0.1)^T\right)$ is $(1.1,\,-1)^T$. To get some idea how good (or bad) this is, note that $\mathbf{f}\left((1.1,\,-1)^T\right) = (-0.869,\,0.1)^T$.

The Inverse Function Theorem tells us when an inverse function exists but not what it is. It is an example of an existence result. Nonetheless we can find the Best Affine Approximation to the inverse function.

Tangent Spaces for Graphs

8. For each of the following scalar-valued functions ϕ, find both a basis for the Tangent Space and the equation of the Tangent Plane to the graph of φ for the given point on the graph. For the latter give your answer in the form "the Tangent plane to φ at $\mathbf{q}$ is the graph of the function $g(\mathbf{x}) = ...$".

i. $\varphi(\mathbf{x}) = 4x^2 + y^2, \qquad \mathbf{q} = (1,-1)^T \in \mathbb{R}^2,$

ii. $\varphi(\mathbf{x}) = \sqrt{9 - x^2 - y^2}, \qquad \mathbf{q} = (2,1)^T \in \mathbb{R}^2,$

iii. $\varphi(\mathbf{x}) = 9 - x^2 - y^2, \qquad \mathbf{p} = (2,-2,1)^T \in G_\varphi,$

iv. $\varphi(\mathbf{x}) = 5/\left(1 + x^2 + 3y^2\right), \qquad \mathbf{p} = (1,-1,1)^T \in G_\varphi.$

Solution *First recall part of the Theory:* From the notes:

• the graph of φ is the image of

$$\mathbf{F}(\mathbf{x}) := \begin{pmatrix} \mathbf{x} \\ \varphi(\mathbf{x}) \end{pmatrix},$$

and a basis for the Tangent Space at $\mathbf{p} = \mathbf{F}(\mathbf{q})$ is given by the columns of $J\mathbf{F}(\mathbf{q})$.

• The Jacobian matrix of F is

$$J\mathbf{F}(\mathbf{x}) = \begin{pmatrix} I \\ J\varphi(\mathbf{x}) \end{pmatrix},$$

where I is the identity matrix.

• The Tangent Plane to graph of φ at $\mathbf{p}$ is the image of the Best Affine Approximation to $\mathbf{F}$ at $\mathbf{q}$.

• The Best Affine Approximation to $\mathbf{F}$ at $\mathbf{q}$ is

$$\begin{aligned} \mathbf{F}(\mathbf{q}) + J\mathbf{F}(\mathbf{q})\,(\mathbf{x} - \mathbf{q}) &= \begin{pmatrix} \mathbf{q} \\ \varphi(\mathbf{q}) \end{pmatrix} + \begin{pmatrix} I \\ J\varphi(\mathbf{q}) \end{pmatrix} (\mathbf{x} - \mathbf{q}) \\ &= \begin{pmatrix} \mathbf{q} \\ \varphi(\mathbf{q}) \end{pmatrix} + \begin{pmatrix} \mathbf{x} - \mathbf{q} \\ J\varphi(\mathbf{q})(\mathbf{x} - \mathbf{q}) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{x} \\ \varphi(\mathbf{q}) + J\varphi(\mathbf{q})(\mathbf{x} - \mathbf{q}) \end{pmatrix}. \end{aligned}$$

That is, the Best Affine Approximation to $\mathbf{F}$ at $\mathbf{q}$ is the graph of the Best Affine Approximation to φ at $\mathbf{q}$.

i. First note that, given $\mathbf{q} = (1,-1)^T \in \mathbb{R}^2$, we are looking for the Tangent Space and plane at the point

$$\mathbf{p} = \begin{pmatrix} \mathbf{q} \\ \varphi(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$$

on the graph. Next, $J\varphi(\mathbf{x}) = (8x^2, 2y)$ so $J\varphi(\mathbf{q}) = (8, -2)$. Thus

$$J\mathbf{F}(\mathbf{q}) = \begin{pmatrix} I_2 \\ J\varphi(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 8 & -2 \end{pmatrix},$$

and hence the columns of this matrix, the vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

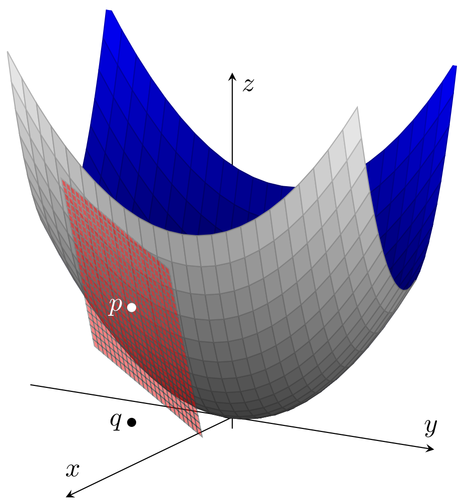
form a basis for the Tangent Space at $\mathbf{p}$.

Also, since $\varphi(\mathbf{q}) = 5$, the Best Affine Approximation to φ at $\mathbf{q}$ is

$$\varphi(\mathbf{q}) + J\varphi(\mathbf{q})(\mathbf{x} - \mathbf{q}) = 5 + \begin{pmatrix} 8 & -2 \end{pmatrix} \begin{pmatrix} x - 1 \\ y + 1 \end{pmatrix} = 8x - 2y - 5.$$

Hence the Tangent Plane to the graph of φ at $\mathbf{p}$ is the graph of the function $g\,(\mathbf{x}) = 8x - 2y - 5$.

Figure for Question 8i:



ii. This time $\mathbf{p} = (2,1,2)^T$ while $\varphi(\mathbf{q}) = 2$. Next

$$J\varphi(\mathbf{x}) = \left(-\frac{x}{\sqrt{9-x^2-y^2}}, -\frac{y}{\sqrt{9-x^2-y^2}} \right),$$

so $J\varphi(\mathbf{q}) = (-1, -1/2)$. Thus

$$J\mathbf{F}(\mathbf{q}) = \begin{pmatrix} I_2 \\ J\varphi(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1/2 \end{pmatrix},$$

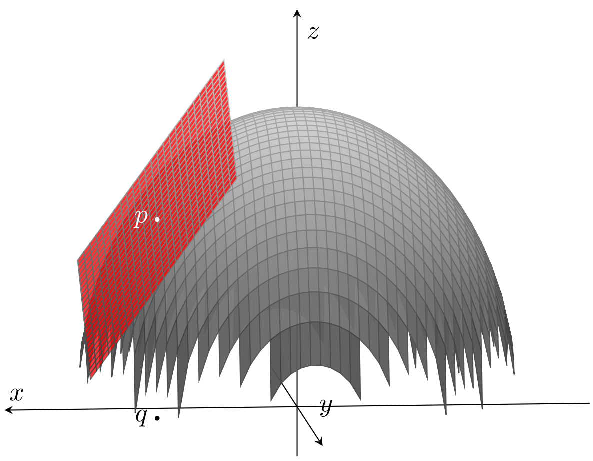
and so $(1,0,-1)^T$ and $(0,1,-1/2)^T$ form a basis for the Tangent Space at $\mathbf{p}$.

The Best Affine Approximation to φ at $\mathbf{q}$ is

$$\varphi(\mathbf{q}) + J\varphi(\mathbf{q})(\mathbf{x}-\mathbf{q}) = 2 + \begin{pmatrix} -1 & -1/2 \end{pmatrix} \begin{pmatrix} x-2 \\ y-1 \end{pmatrix} = -x - \frac{1}{2}y + \frac{9}{2}.$$

Hence the Tangent plane to the graph of φ at $\mathbf{p}$ is the graph of the function $g(\mathbf{x}) = -x - y/2 + 9/2$.

Figure for Question 7ii.



The boundary of the surface is jagged since it is plotted using rectangular coordinates rather than spherical ones.

In the next two parts we are given a point $\mathbf{p}$ *on the graph.* Being *on* the graph means that

$$\mathbf{p} = \begin{pmatrix} \mathbf{q} \\ \varphi(\mathbf{q}) \end{pmatrix},$$

for some $\mathbf{q} \in \mathbb{R}^2$.

iii. Since $\mathbf{p} = (2, -2, 1)^T$ then $\mathbf{q} = (2, -2)^T$ and $\varphi(\mathbf{q}) = 1$. Also

$$J\varphi(\mathbf{x}) = \begin{pmatrix} -2x & -2y \end{pmatrix} \quad \text{in which case} \quad J\varphi(\mathbf{q}) = \begin{pmatrix} -4 & 4 \end{pmatrix}.$$

Thus the columns of

$$\begin{pmatrix} I_2 \\ J\varphi(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -4 & 4 \end{pmatrix},$$

namely $(1, 0, -4)^T$ and $(0, 1, 4)^T$, form a basis for the Tangent Space at $\mathbf{p}$.

The Best Affine Approximation to φ at $\mathbf{q}$ is

$$1 + \begin{pmatrix} -4 & 4 \end{pmatrix} \begin{pmatrix} x-2 \\ y+2 \end{pmatrix} = -4x + 4y + 17.$$

Hence the Tangent Plane to the graph of φ at $\mathbf{p} \in G_\varphi$ is the graph of the function $g(\mathbf{x}) = -4x + 4y + 17$.

iv. Since $\mathbf{p} = (1, -1, 1)^T$ then $\mathbf{q} = (1, -1)^T$ and $\varphi(\mathbf{q}) = 1$. Also

$$\varphi(\mathbf{x}) = \frac{5}{1 + x^2 + 3y^2} \quad \text{implies} \quad J\varphi(\mathbf{x}) = \left(-\frac{10x}{(1 + x^2 + 3y^2)^2}, -\frac{30y}{(1 + x^2 + 3y^2)^2} \right).$$

So $J\varphi(\mathbf{q}) = (-2/5, 6/5)$. Thus the columns of

$$\begin{pmatrix} I_2 \\ J\varphi(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2/5 & 6/5 \end{pmatrix}$$

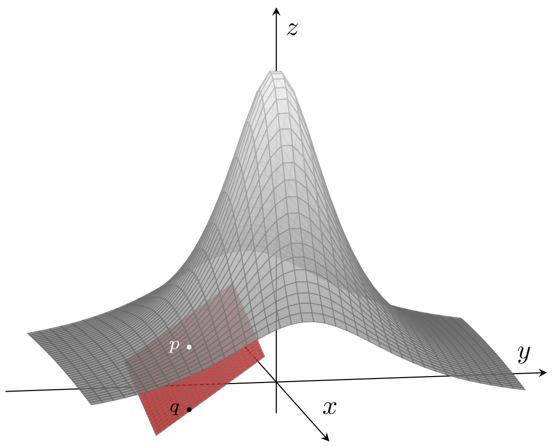
form a basis for the Tangent Space to G_φ at $\mathbf{p}$. We can scale these and give the basis as $(5, 0, -2)^T$ and $(0, 5, 6)^T$.

The Best Affine Approximation to φ at $\mathbf{q}$ is

$$1 + \begin{pmatrix} -\frac{2}{5} & \frac{6}{5} \end{pmatrix} \begin{pmatrix} x-1 \\ y+1 \end{pmatrix} = \frac{-2x + 6y + 13}{5}.$$

Hence the Tangent Plane to the graph of φ at $\mathbf{p} \in G_\varphi$ is the graph of the function $g(\mathbf{x}) = (-2x + 6y + 13)/5$.

Figure for Question 8iv:



9. Repeat Question 8 for the **vector-valued** function

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{pmatrix} xy \\ x^2 + y^2 \end{pmatrix}$$

with $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, at $\mathbf{p} = (2, -1, -2, 5)^T \in G_{\boldsymbol{\phi}}$.

Solution First note that $\mathbf{p} = \left(\mathbf{q}^T, \boldsymbol{\phi}(\mathbf{q})^T\right)^T$ with $\mathbf{q} = (2, -1)^T$. Then

$$J\boldsymbol{\phi}(\mathbf{x}) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix} \quad \text{so} \quad J\boldsymbol{\phi}(\mathbf{q}) = \begin{pmatrix} -1 & 2 \\ 4 & -2 \end{pmatrix}.$$

This matrix is of full rank so the column of

$$\begin{pmatrix} I_2 \\ J\boldsymbol{\phi}(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 4 & -2 \end{pmatrix},$$

that is $(1, 0, -1, 4)^T$ and $(0, 1, 2, -2)^T$, form a basis to the Tangent Space to G_φ at $\mathbf{p}$.

The Best Affine Approximation to φ at $\mathbf{q}$ is

$$\begin{aligned} \varphi(\mathbf{q}) + J\varphi(\mathbf{q})(\mathbf{x} - \mathbf{q}) &= \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x - 2 \\ y + 1 \end{pmatrix} \\ &= \begin{pmatrix} -x + 2y + 2 \\ 4x - 2y - 5 \end{pmatrix}. \end{aligned}$$

Hence the Tangent Plane to the graph of φ at $\mathbf{p} \in G_\varphi$ is the graph of the function

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} -x + 2y + 2 \\ 4x - 2y - 5 \end{pmatrix}.$$

Solutions to Additional Questions 7

9. Explain why

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} xy \\ x^2 - y^2 \end{pmatrix}$$

is invertible in some neighbourhood of $\mathbf{p} = (1,1)^T$.

Calculate the Fréchet derivative of the inverse at $\mathbf{q} = \mathbf{f}(\mathbf{p}) = (1,0)^T$.

Solution The Jacobian matrix at $\mathbf{x} \in \mathbb{R}^2$ is

$$J\mathbf{f}(\mathbf{x}) = \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}.$$

This is invertible for all $\mathbf{x} \neq \mathbf{0}$. In particular the Jacobian matrix is invertible at $\mathbf{p}$ and so, by the Inverse Function Theorem, $\mathbf{f}$ is locally invertible near $\mathbf{p}$.

Also, from the Inverse Function Theorem, if $\mathbf{g}$ is the inverse of f then

$$J\mathbf{g}(\mathbf{q}) = J\mathbf{f}(\mathbf{p})^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$

The question asked for the Fréchet derivative:

$$d\mathbf{g}_{\mathbf{q}}(\mathbf{t}) = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2s+t \\ 2s-t \end{pmatrix},$$

for $\mathbf{t} = (s,t)^T \in \mathbb{R}^2$.

10. At what points are the functions below, from $\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$ to $\mathbb{R}^3$, invertible?

i.

$$\mathbf{f}_1(\mathbf{x}) = \left(x^2 + 2xz,\ 2\sqrt{y},\ z^2 + xz\right)^T,$$

ii

$$\mathbf{f}_2(\mathbf{x}) = \left(x^2 + xz,\ 2\sqrt{y},\ z^2 + xz\right)^T,$$

iii.

$$\mathbf{f}_3(\mathbf{x}) = \left(x^2 + xz/2,\ 2\sqrt{y},\ z^2 + xz\right)^T.$$

Solution i. The Jacobian matrix is

$$J\mathbf{f}_1(\mathbf{x}) = \begin{pmatrix} 2x+2z & 0 & 2x \\ 0 & \frac{1}{\sqrt{y}} & 0 \\ z & 0 & 2z+x \end{pmatrix}.$$

The determinant is
$$\frac{2}{\sqrt{y}}\left((x+z)^2+z^2\right).$$
This is only zero if $z=0$ and $x=0$. Hence $\mathbf{f}_1$ is invertible for all $\mathbf{x}$ except for the straight line $(0,y,0)^T : y>0$.

ii. It can be checked that
$$\det J\mathbf{f}_2(\mathbf{x})=\frac{2}{\sqrt{y}}(x+z)^2$$

Hence $\mathbf{f}_2$ is invertible for all $\mathbf{x}$ except for the plane $(x,y,-x)^T : x\in\mathbb{R}, y>0$.

iii. It can be checked that
$$\det J\mathbf{f}_3(\mathbf{x})=\frac{1}{\sqrt{y}}\left(2(x+z)^2-z^2\right)$$

Hence $\mathbf{f}_3$ is invertible for all $\mathbf{x}$ except for the two planes $\left(x,y,-\sqrt{2}x/\left(\sqrt{2}-1\right)\right)^T$ and $\left(x,y,-\sqrt{2}x/\left(\sqrt{2}+1\right)\right) : x\in\mathbb{R}, y>0$.

11. Consider the surface in $\mathbb{R}^4$:
$$\left\{\begin{pmatrix} yz\\ xz\\ xy\\ xyz\end{pmatrix} : \mathbf{x}\in\mathbb{R}^3\setminus\{\mathbf{0}\}\right\}=\left\{\begin{pmatrix}\mathbf{f}(\mathbf{x})\\ xyz\end{pmatrix} : \mathbf{x}\in\mathbb{R}^3\setminus\{\mathbf{0}\}\right\},$$

where $\mathbf{f}(\mathbf{x})=(yz,xz,xy)^T$. The point $\mathbf{p}=(-2,2,-1,-2)^T$ on the surface is the image of $\mathbf{a}=(1,-1,2)^T$.

I have introduced the function $\mathbf{f}$ since it was the subject of Question 2ii where it was shown that $\mathbf{f}$ has an inverse in an open set containing $\mathbf{a}$. That is there exists $V:\mathbf{a}\in V\subseteq\mathbb{R}^3$ and $W\subseteq\mathbb{R}^3$ such that $\mathbf{f}:V\to W$ has an inverse, $\mathbf{g}$ say. For a general $\mathbf{x}\in V$ write $\mathbf{s}=\mathbf{f}(\mathbf{x})$, so $\mathbf{s}\in W\subseteq\mathbb{R}^3$. Also, let $\mathbf{b}=\mathbf{f}(\mathbf{a})=(-2,2,-1)^T$.

Since $\mathbf{g}$ is the inverse of $\mathbf{f}$ we have $\mathbf{x}=\mathbf{g}(\mathbf{s})$. This means the coordinates of $\mathbf{x}$, i.e. x,y and z, can be written as functions of $\mathbf{s}$. Thus xyz is a function of $\mathbf{s}$, i.e. $xyz=\phi(\mathbf{s})$, say. Then our surface is locally a graph:
$$\left\{\begin{pmatrix}\mathbf{f}(\mathbf{x})\\ xyz\end{pmatrix} : \mathbf{x}\in V\right\}=\left\{\begin{pmatrix}\mathbf{s}\\ \phi(\mathbf{s})\end{pmatrix} : \mathbf{s}\in W\right\}. \qquad (3)$$

We know little about $\phi : W\to\mathbb{R}$, though you can check that $\phi(\mathbf{b})=-2$. We can, though, calculate the derivative at $\mathbf{b}$.

Question Calculate $J\phi(\mathbf{b})$.

Hint Write $k(\mathbf{x}) = xyz$ for $\mathbf{x} \in V$, express ϕ as a convolution of $\mathbf{g}$ and k and apply the Chain Rule .

Solution Write $k(\mathbf{x}) = xyz$ for $\mathbf{x} \in V$. Then for $\mathbf{s} \in W$ we have, from (3) ,

$$\phi(\mathbf{s}) = xyz = k(\mathbf{x}) = k(\mathbf{g}(\mathbf{s}))\,.$$

Apply the Chain Rule, so $J\phi(\mathbf{s}) = Jk(\mathbf{g}(\mathbf{s}))\, J\mathbf{g}(\mathbf{s})$. Choose $\mathbf{s} = \mathbf{b}$:

$$J\phi(\mathbf{b}) = Jk(\mathbf{g}(\mathbf{b}))\, J\mathbf{g}(\mathbf{b}) = Jk(\mathbf{a})\, J\mathbf{g}(\mathbf{b})\,, \tag{4}$$

since $\mathbf{a} = \mathbf{g}(\mathbf{b})$. The second matrix on the right hand side of (4) has been calculated in Question 2ii:

$$\begin{aligned} J\mathbf{g}(\mathbf{b}) &= J\mathbf{g}(\mathbf{f}(\mathbf{a})) = \frac{1}{2abc}\begin{pmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{pmatrix}_{\mathbf{a}=(1,-1,2)^T} \\ &= \frac{1}{4}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}. \end{aligned}$$

The first matrix on right hand side of (4) is

$$Jk(\mathbf{a}) = (yz,\, xz,\, xy)_{\mathbf{x}=(1,-1,2)^T} = \begin{pmatrix} -2, & 2, & -1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} J\phi(\mathbf{b}) &= \frac{1}{4}(-2, 2, -1)\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix} \\ &= \frac{1}{2}(1, -1, 2)\,. \end{aligned}$$

Note this answer looks coincidentally like $\mathbf{a}^T/2$. It is not a coincidence, in general for $\mathbf{a} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ and $\mathbf{b} = \mathbf{f}\,(\mathbf{a})$,

$$\begin{aligned} J\phi\,(\mathbf{b}) &= \frac{1}{2abc}(bc,\, ac,\, ab)\begin{pmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{pmatrix} \\ &= \frac{1}{2}(a,\, b,\, c) = \frac{1}{2}\mathbf{a}^T. \end{aligned}$$

12. Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{f}\big((u,v)^T\big) = (u^3 + uv + v^3,\ u^2 - v^2)^T$. Show that $\mathbf{f}$ locally invertible at $\mathbf{a} = (1,1)^T$.

What is the Best Affine Approximation to the **inverse** function near $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (3,0)^T$?

What approximation does this give to $\mathbf{f}^{-1}(\mathbf{b}')$ where $\mathbf{b}' = (3.1, -0.2)^T$?

Solution The Jacobian matrix is

$$J\mathbf{f}(\mathbf{a}) = \begin{pmatrix} 3u^2 + v & u + 3v^2 \\ 2u & -2v \end{pmatrix}_{(1,1)^T} = \begin{pmatrix} 4 & 4 \\ 2 & -2 \end{pmatrix}.$$

The matrix is of full-rank and so, by the Inverse Function Theorem, $\mathbf{f}$ is invertible in some open set containing $\mathbf{a}$ (i.e. it is locally invertible). From the Chain Rule applied to $\mathbf{f} \circ \mathbf{f}^{-1} = \mathbf{id}$ we deduce that the Jacobian of the inverse is the inverse of the Jacobian. That is, with $\mathbf{b} = \mathbf{f}(\mathbf{a})$, the Jacobian of the inverse is

$$J\mathbf{f}^{-1}(\mathbf{b}) = J\mathbf{f}(\mathbf{a})^{-1} = \frac{1}{8}\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}.$$

Then the Best Affine Approximation to $\mathbf{f}^{-1}$ near $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (3,0)^T$ is, as a function of $\mathbf{x} = (x,y)^T \in \mathbb{R}^2$,

$$\begin{aligned} \mathbf{f}^{-1}(\mathbf{b}) + J\mathbf{f}^{-1}(\mathbf{b})(\mathbf{u} - \mathbf{b}) &= \mathbf{a} + J\mathbf{f}(\mathbf{a})^{-1}(\mathbf{u} - \mathbf{f}(\mathbf{a})) \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x - 3 \\ y \end{pmatrix} \\ &= \frac{1}{8}\begin{pmatrix} 5 + x + 2y \\ 5 + x - 2y \end{pmatrix}. \end{aligned}$$

The approximation this gives to $\mathbf{f}^{-1}\left((3.1, -0.2)^T\right)$ is $(0.9625, 1.0625)^T$. To get some idea how good (or bad) this is, note that $\mathbf{f}\left((0.9625, 1.0625)^T\right) = (3.113..., -0.2025)^T$.

13. *For additional practice* For each function, find the Best Affine Approximation at the given point. With $\mathbf{s} = (s,t)^T \in \mathbb{R}^2$,

i. $\mathbf{f}(\mathbf{s}) = (t\cos s, t\sin s, t)^T$, $\mathbf{q} = (\pi/2, 2)^T$,

ii. $\mathbf{f}(\mathbf{s}) = (t^2 \cos s, t^2, t^2 \sin s)^T, \mathbf{q} = (0,1)^T$,

Solution i. The Best Affine Approximation is

$$\begin{aligned} \mathbf{f}(\mathbf{q}) + J\mathbf{f}(\mathbf{q})\,(\mathbf{s}-\mathbf{q}) &= \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s - \pi/2 \\ t-2 \end{pmatrix} \\ &= \begin{pmatrix} -2s+\pi \\ t \\ t \end{pmatrix}. \end{aligned}$$

Note that this is the same set as

$$\left\{ \begin{pmatrix} x \\ y \\ y \end{pmatrix} : (x,y)^T \in \mathbb{R}^2 \right\}.$$

The plane can also be given simply as the level set $\{\mathbf{x} \in \mathbb{R}^3 : z = y\}$.

ii. The Best Affine Approximation is

$$\begin{aligned} \mathbf{f}(\mathbf{q}) + J\mathbf{f}(\mathbf{q})\,(\mathbf{s}-\mathbf{q}) &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s - 0 \\ t-1 \end{pmatrix} \\ &= \begin{pmatrix} -1+2t \\ -1+2t \\ s \end{pmatrix}. \end{aligned}$$

Note that this is the same set as

$$\left\{ \begin{pmatrix} x \\ x \\ z \end{pmatrix} : (x,z)^T \in \mathbb{R}^2 \right\}.$$

The plane can also be given simply as the level set $\{\mathbf{x} \in \mathbb{R}^3 : x = y\}$.

14 In each of the following examples, find both a basis for the Tangent Space and the equation of the Tangent Plane to the graph of ϕ for the given point:

i. $\phi(\mathbf{x}) = x\,(x+y)$, at $\mathbf{q} = (2,-1)^T \in \mathbb{R}^2$,

ii. $\phi(\mathbf{x}) = (x-1)^2 + y^2 \quad$ at $\mathbf{q} = (0,2)^T \in \mathbb{R}^2$,

iii. $\phi(\mathbf{x}) = \sin(xy^2z^3) \quad$ at $\mathbf{q} = (\pi, 1, -1)^T$.

iv.
$$\boldsymbol{\phi}(\mathbf{x}) = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}$$
at $\mathbf{q} = (2,-3)^T$, and then again at $\mathbf{q} = (2,1)^T$,

v.
$$\boldsymbol{\phi}(\mathbf{x}) = \begin{pmatrix} xy \\ yz \end{pmatrix}$$
at $\mathbf{q} = (1,-1,2)^T$.

Hint Most of these functions have appeared in previous questions. It may save time to quote the results already proved.

Solution i. Look back to the answer to Question 6 to find that the Best Affine Approximation to ϕ at $\mathbf{q}$ is $g(\mathbf{x}) = 3x + 2y - 4$. The Tangent Plane to ϕ at $\mathbf{q}$ is the graph of g :

$$\begin{pmatrix} \mathbf{x} \\ g(\mathbf{x}) \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix},$$

and from this we get that $(1,0,3)^T$ and $(0,1,1)^T$ form a basis for the Tangent Space.

ii. First calculate
$$J\phi(\mathbf{q}) = \nabla\phi(\mathbf{q})^T = (-2,4).$$
Then the Best Affine Approximation to ϕ at $\mathbf{p}$ is

$$\begin{aligned} g(\mathbf{x}) &= \phi(\mathbf{q}) + J\phi(\mathbf{q})(\mathbf{x}-\mathbf{q}) = 5 + (-2,4)\begin{pmatrix} x-0 \\ y-2 \end{pmatrix} \\ &= -2x + 4y - 3. \end{aligned}$$

The Tangent Plane to the graph of ϕ at $\mathbf{q}$ is the graph of $g(\mathbf{x}) = -2x+4y-3$. As in part i, from this we get a basis for the Tangent Space of $(1,0,-2)^T$ and $(0,1,4)^T$.

iii. Look back to the answer to Question 2 Sheet 5 where the gradient $\nabla\phi(\mathbf{q}) = (1, 2\pi, -3\pi)^T$ was calculated. Then the Tangent Plane to the graph of ϕ at $\mathbf{q}$ is the graph of

$$\begin{aligned} g(\mathbf{x}) &= \phi(\mathbf{q}) + J\phi(\mathbf{q})(\mathbf{x}-\mathbf{q}) = \sin(-\pi) + (1, 2\pi, -3\pi)\begin{pmatrix} x-\pi \\ y-1 \\ z+1 \end{pmatrix} \\ &= x + 2\pi y - 3\pi z - 6\pi. \end{aligned}$$

A basis for the Tangent Space is then $(1,0,0,1)^T$, $(0,1,0,2\pi)^T$ and $(0,0,1,-3\pi)^T$.

iv. This function was the subject of Question 5ii, Sheet 5, but directly we have

$$J\boldsymbol{\phi}(\mathbf{q}) = \left.\begin{pmatrix} y^2 & 2xy \\ 2xy & x^2 \end{pmatrix}\right|_{\mathbf{x}=\mathbf{q}} = \begin{pmatrix} 9 & -12 \\ -12 & 4 \end{pmatrix}$$

when $\mathbf{q} = (2,-3)^T$. Thus the Tangent Plane to the graph of $\boldsymbol{\phi}$ at $\mathbf{q}$ is the graph of

$$\begin{aligned} \mathbf{g}(\mathbf{x}) &= \boldsymbol{\phi}(\mathbf{q}) + J\boldsymbol{\phi}(\mathbf{q})(\mathbf{x}-\mathbf{q}) = \begin{pmatrix} 18 \\ -12 \end{pmatrix} + \begin{pmatrix} 9 & -12 \\ -12 & 4 \end{pmatrix}\begin{pmatrix} x-2 \\ y+3 \end{pmatrix} \\ &= \begin{pmatrix} 9x-12y-36 \\ -12x+4y+24 \end{pmatrix}. \end{aligned}$$

A basis for the Tangent Space is then $(1,\ 0,\ 9,\ -12)^T$ and $(0,\ 1,\ -12,\ 4)^T$.

When $\mathbf{q} = (2,1)^T$,

$$J\boldsymbol{\phi}(\mathbf{q}) = \begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix}.$$

Thus the Tangent Plane to the graph of $\boldsymbol{\phi}$ at $\mathbf{q}$ is the graph of

$$\begin{aligned} \mathbf{g}(\mathbf{x}) &= \boldsymbol{\phi}(\mathbf{q}) + J\boldsymbol{\phi}(\mathbf{q})(\mathbf{x}-\mathbf{q}) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x-2 \\ y-1 \end{pmatrix} \\ &= \begin{pmatrix} x+4y-4 \\ 4x+4y-8 \end{pmatrix}. \end{aligned}$$

A basis for the Tangent Space is then $(1,0,1,4)^T$ and $(0,1,4,4)^T$.

v. The Jacobian matrix is

$$J\boldsymbol{\phi}(\mathbf{q}) = \left.\begin{pmatrix} y & x & 0 \\ 0 & z & y \end{pmatrix}\right|_{\mathbf{x}=\mathbf{q}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix},$$

when $\mathbf{q} = (1, -1, 2)^T$. Thus the Tangent Plane to the graph of $\boldsymbol{\phi}$ at $\mathbf{q}$ is the graph of

$$\begin{aligned}
\mathbf{g}(\mathbf{x}) &= \boldsymbol{\phi}(\mathbf{q}) + J\boldsymbol{\phi}(\mathbf{q})(\mathbf{x} - \mathbf{q}) = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x - 1 \\ y + 1 \\ z - 2 \end{pmatrix} \\
&= \begin{pmatrix} -x + y + 1 \\ 2y - z + 2 \end{pmatrix}.
\end{aligned}$$

In fact, the graph of $\mathbf{g}(\mathbf{x})$ consists of the points

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ -x + y + 1 \\ 2y - z + 2 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

A basis for the Tangent Space is then $(1, 0, 0, -1, 0)^T$, $(0, 1, 0, 1, 2)^T$ and $(0, 0, 1, 0, -1)^T$.

15. Find the Tangent plane to the graph of

$$\phi(\mathbf{x}) = \frac{x^3 - y^3 + 1}{(x + y)^4 + 1},$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, at the point $(2, -1, 5)^T$ *on* the graph.

Hint Multiply up before you differentiate.

Solution The point $\mathbf{p} = (2, -1, 5)^T$ *on* the graph is $\left(\mathbf{q}^T, \phi(\mathbf{q})\right)^T$ where $\mathbf{q} = (2, -1)^T$. Following the hint write

$$\left((x + y)^4 + 1\right) \phi(\mathbf{x}) = x^3 - y^3 + 1. \tag{5}$$

Differentiate (5) w.r.t. x to get

$$4 (x + y)^3 \phi(\mathbf{x}) + \left((x + y)^4 + 1\right) \frac{\partial \phi}{\partial x}(\mathbf{x}) = 3x^2.$$

Choose $\mathbf{x} = \mathbf{q}$ when

$$4\phi(\mathbf{q}) + 2\frac{\partial \phi}{\partial x}(\mathbf{q}) = 12.$$

But $\phi(\mathbf{q}) = 5$ so this rearranges to $\partial\phi(\mathbf{q})/\partial x = -4$.

Differentiate (5) w.r.t. y to get

$$4(x+y)^3 \phi(\mathbf{x}) + \left((x+y)^4 + 1\right) \frac{\partial \phi}{\partial y}(\mathbf{x}) = -3y^2.$$

Choose $\mathbf{x} = \mathbf{q}$ when

$$4\phi(\mathbf{q}) + 2\frac{\partial \phi}{\partial y}(\mathbf{q}) = -3 \quad \text{and so} \quad \frac{\partial \phi}{\partial y}(\mathbf{q}) = -\frac{23}{2}.$$

Hence

$$J\phi(\mathbf{q}) = \left(-4, -\frac{23}{2}\right).$$

The Tangent plane to a graph of a function is given by the graph of the Best Affine Approximation to that function. In this case the Best Affine Approximation to ϕ near $\mathbf{q}$ is

$$\begin{aligned} \phi(\mathbf{q}) + J\phi(\mathbf{q})(\mathbf{x} - \mathbf{q}) &= 5 + \left(-4, -\frac{23}{2}\right)\begin{pmatrix} x-2 \\ y+1 \end{pmatrix} \\ &= -\frac{8x + 23y - 3}{2} \end{aligned}$$

That is, the Tangent plane is $8x + 23y + 2z = 3$. As a graph

$$\left\{ \begin{pmatrix} x \\ y \\ (-8x - 23y + 3)/2 \end{pmatrix} : (x,y)^T \in \mathbb{R}^2 \right\}.$$

16 Find the Tangent plane to the graph of

$$\phi(\mathbf{x}) = \frac{x^2 y + 2xy^2}{1 + x^2 + y^2}$$

where $\mathbf{x} = (x,y)^T \in \mathbb{R}^2$, at the point $\mathbf{q} = (1,2)^T$.

Solution Follow the method of the previous solution and multiply up to find

$$J\phi(\mathbf{q}) = \nabla\phi(\mathbf{q})^T = \frac{1}{18}(26, 7).$$

The Tangent plane to a graph of a function is given by the graph of the Best Affine Approximation to that function. In this case the Best Affine Approximation to ϕ near $\mathbf{q}$ is

$$\phi(\mathbf{q}) + J\phi(\mathbf{q})(\mathbf{x} - \mathbf{q}) = \frac{5}{3} + \frac{1}{18}(26, 7)\begin{pmatrix} x-1 \\ y-2 \end{pmatrix} = \frac{26x + 7y - 10}{18}.$$